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Properties and possibilities of quantum shapelets

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Abstract

Quantum shapelets arise as the solution of a d-dimensional harmonic oscillator or D-dimensional Coulomb problem and may be obtained by requiring scale-space invariance. These functions have application to image processing in conventional or quantum contexts. We recall the scale-space-based derivation of shapelets and present novel properties of these functions, including integral relations, infinite series and finite convolution sums. Many of these relations also have application to the combinatorics of zero-dimensional quantum field theory.

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1. Introduction

The concept of scale spaces [2, 25] has proved very useful for image processing applications. In scale-space theory, one embeds an image into a continuous family of gradually smoother versions of it. The time t acts as a parameter for this, with the original image corresponding to $t = 0$. Increasing the scale should simplify the image without creating spurious structure [18]. For instance, in viewing a facial image at coarser scales, it would be undesirable to have artificial features appearing. A scale space introduces a hierarchy of image features, and can provide an important process in going from a pixel-level description to a semantical image description [29].

Here we report a number of intriguing connections between quantum mechanics, diffusion, scale-space invariance, and image processing. These are based upon the properties of quantum shapelets that are the solutions of the simple harmonic oscillator in various dimensions d. We recall that such solutions are intimately related to the solutions of hydrogenic atoms (the Coulomb problem) in dimension D [20]. One example is the four-dimensional harmonic oscillator solution Kustaanheimo–Steifel transformed to the three-dimensional Kepler–Coulomb problem solution [20].

Our investigation shows that linear scale space will be implementable in either a hybrid or purely quantum environment. In the next section, we describe this and recall how quantum shapelets may be derived on the basis of scale-space requirements [18].

Gaussian derivative kernels are very useful in current image processing for filtering, convolution and other applications. A particular example is provided by astronomical image processing, including Hubble space telescope image compression and reconstruction, effects of gravitational lensing and image de-projection [6, 7, 11, 21, 22, 26]. Therefore our discussion and analytic techniques may benefit image processing both in current conventional computing environments and on future quantum processors. Shapelets are particularly useful when objects of interest within an image are well localized in space.

Quantum shapelets possess a large number of known useful properties, including invariance under Fourier transform and change of scale, compact representations for convolution and providing a basis for coherent states. Still, these properties are not exhausted and we present some analytic methods that complement existing relations and identities.

Besides the quantum Fourier transform, the quantum wavelet transform is known for cases including the Haar and Daubechies $D^{(4)}$ wavelets [15]. The quantum circuits for these use tensor products of 2×2 matrices together with permutation or swap operators. Wavelets are generally most useful when a signal or image possesses abrupt changes in signal strength or pixel intensity. Wavelets are well able to represent discontinuities since they are based upon the translations and dilations of a given function. As a by-product of our research, we are able to quickly present the answer to a problem very recently posed in the search for generalizations of the Mexican hat wavelet [14]. This result is given briefly in the final remark of the appendix.

In recent years the combinatorics of the general boson normal ordering problem have been developed [9]. In such theory, there is a strong interplay between bosonic coherent states, the properties of generalized Bell and Stirling numbers, generating functions and hierarchical Dobinski-type relations [9]. Much of the generating function-based results that we develop are then relevant to combinatorial zero-dimensional quantum field theory via boson normal ordering [10]. In zero-dimensional field theory, all Feynman integrals are unity and the generating function-like relations provide a means of finding the number of Feynman-like graphs.

The formulation of the simple harmonic oscillator problem in terms of raising and lowering operators is well known and need not be recounted. In the context of quantum field theory, the Hamiltonian is handily written in terms of the number operator $a^\dagger a$ where $a^\dagger$ is the creation operator and a is the annihilation operator, satisfying the boson commutation relation $[a, a^\dagger] = 1$. For a radial problem, for instance, $a = \mathrm{d}/\mathrm{d}r$ and $a^\dagger = r$ provide a formal representation on a suitable space of functions. The action of $(r\,\mathrm{d}/\mathrm{d}r)^n$ is then analogous to the normal ordering problem for writing $(a^\dagger a)^n$ with all the annihilators on the right. A coherent state $|z\rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} (z^n/\sqrt{n!})|n\rangle$, where $a^\dagger a|n\rangle = n|n\rangle$, is then seen to be in a ready form for exponential generating functions to be applied.

After describing the origin of shapelets from scale-space ideas in the next section, we give an example of a polar shapelet integral relation useful in current image processing. We then extend some generating function relations for polar shapelets and present new infinite sums and finite convolution series. This may also be considered a contribution to special function theory, as we expand the known properties of the associated Laguerre polynomials. In brief concluding remarks, we indicate directions for future research.

2. Scale space and shapelets

The work of Lions *et al* [2] has shown that partial differential equations are the suitable framework for scale spaces, and the oldest, simplest, and probably most studied version of scale space corresponds to a linear diffusion process. The fundamental solution

(Greens function) for a linear heat or diffusion equation is a Gaussian function with standard deviation proportional to the square root of the time. The solution of the linear diffusion equation can be given as the convolution of the initial data (image) with this Gaussian function, and this gives linear scale space.

Derivatives of a Gaussian kernel are Hermite polynomials times this Gaussian kernel and these result from scale-space considerations in the following way. One may consider the problem of deriving linear operators from the scale-space representation that are invariant under scaling transformations [18]. Starting from a certain ansatz, $\psi_n = \varphi_n g$, where g is a Gaussian function, and demanding that ψ_n satisfy the diffusion equation, it follows that φ_n must satisfy the Schrödinger equation (SE)

$$\nabla^2\varphi + [(2n+d) - x^2]\varphi = 0, \tag{1}$$

where $d = 1, 2$, or 3. This is the time-independent SE of the quantum mechanical simple harmonic oscillator. When Koenderink and van Doorn proposed such solutions for image processing use, they termed them 'ripples' [18]. Other investigators in astrophysics have later applied the term 'shapelets' [21, 26].

Quantum shapelets will be available if quantum processors are realized. For a purely quantum processor, this is because any quantum computer is fully capable of simulating any other. In particular, a quantum computer will be readily capable of solving the harmonic oscillator problem and generating its eigenstates. Indeed, for an example of a quantum computer based upon this particular problem, one may consult the well-known book of Nielsen and Chuang [23] for such a model. In a hybrid quantum-classical context, shapelets will be available since there is an efficient quantum lattice gas algorithm for the Schrödinger equation, even in the multiparticle case [31]. In this algorithm, applications of a unitary collision operator alternate with streaming operations.

The explicit representation of quantum shapelets is in terms of products of Gaussian functions and Hermite (H_n) or Laguerre polynomials, depending upon the spatial dimension and the coordinate system used. Omitting normalization factors, Cartesian shapelet basis functions for two-dimensional image applications are given by

$$\phi_{n_1n_2}(x, y) \propto H_{n_1}\left(\frac{x}{\beta}\right) H_{n_2}\left(\frac{y}{\beta}\right) \exp[-(x^2+y^2)/2\beta^2], \tag{2a}$$

and polar shapelet basis functions are given by

$$\chi_{nm}(r, \theta) \propto r^{|m|} L^{|m|}_{(n-|m|)/2}\left(\frac{r^2}{\beta^2}\right) \exp[-r^2/2\beta^2] \exp(\mathrm{i}m\theta), \tag{2b}$$

where L_n^α is an associated Laguerre polynomial. For the quantum mechanical isotropic harmonic oscillator problem, $\beta^2 = \hbar/\sqrt{km}$, where m is the particle mass and k is the spring constant, while for shapelet image processing, β becomes a width parameter. The quantum numbers n_i correspond to the energy levels for Cartesian shapelets as $E_{n_1n_2} = (n_1+n_2+1)\hbar\omega$, where $\omega = \sqrt{k/m}$ and $n_1, n_2 = 0, 1, 2, \ldots$. For polar shapelets, m corresponds to the angular momentum and the energies are given by $E_{nm} = (2n + |m| + 1)\hbar\omega$ with $n = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$. The important rotationally invariant basis states have $m = 0$.

The Hermite and Laguerre polynomials are closely related to one another, and to the confluent hypergeometric function ${}_1F_1$. We recall that

$$\binom{n+\alpha}{n} {}_1F_1(-n, \alpha+1, x) = L_n^\alpha(x) = \frac{(-x)^n}{n!} {}_2F_0\left(-n, -n-\alpha; .; -\frac{1}{x}\right), \tag{3}$$

where the latter form follows by reordering of the series for the hypergeometric function. From Kummer's first transformation [3] we have ${}_1F_1(j, \alpha+1, x) = \binom{n-1}{n-\alpha-1}^{-1} \mathrm{e}^x L^\alpha_{j-\alpha-1}(-x)$.

3. An object shape measure

From a shapelet decomposition, the shapelet coefficients may be used to classify object morphologies within an image. In the context of image processing for classifying galaxies, measures of concentration, asymmetry and clumpiness have been shown to correlate with evolutionary type, galaxy merger history and star formation rates, respectively [22]. In this section, we illustrate the shapelet calculus with the concentration index, and relegate to the appendix an integral evaluation useful for the convolution of Cartesian shapelets. We then generalize our integration result to one useful in the study of convolution equation systems on the Heisenberg group H_n [5].

A concentration index has been defined for astronomical images based upon certain percentages of an object's total flux. For images of galaxies, the concentration correlates well with their Hubble type and mass [8, 22]. In calculating the concentration, it is necessary to integrate an object's radial profile, and in so doing Massey and Refregier introduced the integral ([22], equation (47))

$$I_n \equiv \frac{1}{2\beta^2}\int_0^R L_{n/2}\left(\frac{r^2}{\beta^2}\right)\exp(-r^2/2\beta^2) r\,\mathrm{d}r, \tag{4}$$

where β is the scale parameter. We determine I_n alternatively to [22]. In fact, our independent derivation aided in presenting the result there [22]. We have

Proposition 1. *For $y \equiv R^2/\beta^2$ and n a nonnegative even integer, we have*

$$I_n(y) = \frac{(-1)^{n/2}}{2} - \frac{1}{2}\,\mathrm{e}^{-y/2}L_{n/2}(y) - (-1)^{n/2}\,\mathrm{e}^{-y/2}\sum_{j=0}^{n/2-1}(-1)^j L_j(y). \tag{5}$$

Proof of equation (5). By a change of variable and the use of a known summation identity for $L_k(\tau x)$ [4, 13] we have

$$\begin{aligned} I_n(y) &= \frac{1}{2}\int_0^{y/2} L_{n/2}(2w)\,\mathrm{e}^{-w}\,\mathrm{d}w \\ &= \frac{1}{2}\sum_{j=0}^{n/2}\binom{n/2}{j}2^j(-1)^{n/2-j}\int_0^{y/2} L_j(w)\,\mathrm{e}^{-w}\,\mathrm{d}w, \end{aligned} \tag{6}$$

which exhibits that I_n depends only upon the ratio y. By using a tabulated integral [16] we find that

$$\begin{aligned} \int_0^y L_j(w)\,\mathrm{e}^{-w}\,\mathrm{d}w &= \left(\int_0^\infty - \int_y^\infty\right) L_j(w)\,\mathrm{e}^{-w}\,\mathrm{d}w \\ &= \delta_{j0} - \mathrm{e}^{-y}[L_j(y) - L_{j-1}(y)], \end{aligned} \tag{7}$$

where δ_{jk} is the Kronecker symbol and orthogonality of the Laguerre polynomials was used. By applying equation (7) to equation (6), and again using the identity for $L_j(2x)$, we have

$$I_n(y) = \frac{(-1)^{n/2}}{2} - \frac{1}{2}\,\mathrm{e}^{-y/2}L_{n/2}(y) + \frac{(-1)^{n/2}}{2}\,\mathrm{e}^{-y/2}\sum_{j=1}^{n/2}\binom{n/2}{j}2^j(-1)^j L_{j-1}(y/2). \tag{8}$$

We next substitute the expression [13, 4] $L_j(y/2) = 2^{-j}\sum_{\ell=0}^{j}\binom{j}{\ell}L_\ell(y)$ into this equation. We reorder the resulting double sum and apply the orthogonality of the binomial coefficients, thus yielding equation (5). An equivalent of this result is obtained in [22] by iterated integration by parts.

The evaluation of equation (5) for different values of y enables the determination of the concentration index from equations (46) and (59) of [22]. Accordingly, we suggest that equation (5) may be convenient for some applications.

The following function arises in the investigation of the injectivity of the Pompeiu transform in the Heisenberg group [5] (p 219):

$$\Xi_m^{n-1}(x) \equiv \int_0^x \mathrm{e}^{-t/2} t^{n-1} L_m^{n-1}(t)\,\mathrm{d}t, \qquad x \geqslant 0. \tag{9}$$

The technique used for proposition 1 may be extended to[1]

Proposition 2. *For $x \geqslant 0$ and m a positive integer, we have*

$$\Xi_m^{n-1}(x) = 2^n(-1)^m \frac{(m+n-1)!}{m!}\left[1 - \mathrm{e}^{-x/2}\sum_{\ell=0}^{n-1}\frac{1}{\ell!}\left(\frac{x}{2}\right)^\ell \right.$$
$$\left. - 2\,\mathrm{e}^{-x/2}\left(\frac{x}{2}\right)^n \sum_{\ell=0}^{m-1}\frac{(-1)^\ell \ell!}{(\ell+n)!} L_\ell^n(x)\right]. \tag{10}$$

Proof of equation (10). With a simple change of variable and re-expression of $L_m^{n-1}(2w)$ [13, 4] we have

$$\Xi_m^{n-1}(x) = 2^n(-1)^m \sum_{p=0}^m \binom{m+n-1}{m-p} 2^p(-1)^p \int_0^{x/2} L_p^{n-1}(w)\,\mathrm{e}^{-w} w^{n-1}\,\mathrm{d}w, \tag{11}$$

where

$$\int_0^y L_p^{n-1}(w)\,\mathrm{e}^{-w} w^{n-1}\,\mathrm{d}w = \left(\int_0^\infty - \int_y^\infty\right) L_p^{n-1}(w)\,\mathrm{e}^{-w} w^{n-1}\,\mathrm{d}w$$
$$= (n-1)!\delta_{p0} + \frac{1}{p}\,\mathrm{e}^{-y} y^n L_{p-1}^n(y), \qquad p \geqslant 1. \tag{12}$$

In equation (12), orthogonality of the associated Laguerre polynomials has been used, together with their Rodrigues-type formula [16, 3] in order to evaluate the remaining integral. For the $p = 0$ case, we have for $n = 1, 2, \ldots$

$$\int_y^\infty \mathrm{e}^{-w} w^{n-1}\,\mathrm{d}w = \Gamma(n, y) = (n-1)!\,\mathrm{e}^{-y}\sum_{m=0}^{n-1}\frac{y^m}{m!}, \tag{13}$$

where $\Gamma(x, y)$ is the incomplete Gamma function [16, 3]. We next apply equations (12) and (13) to equation (11) and substitute $L_{p-1}^n(x/2) = 2^{1-p}\sum_{\ell=1}^p \binom{p+n-1}{p-\ell} L_{\ell-1}^n(x)$. Then reordering the resulting double sum and summing the binomial coefficients yields equation (10).

Remark. The last term on the right side of equation (10) contains $\mathrm{e}^{-x/2}$ multiplying a polynomial of degree $n + m - 1$. This equation gives the expected reduction as $x \to \infty$ (see footnote 1). By a change of variable, we have

$$\Xi_m^{n-1}(x) = \frac{2}{\beta^{2n}}\int_0^{\beta x^{1/2}} r^{2n-1} L_m^{n-1}\left(\frac{r^2}{\beta^2}\right)\mathrm{e}^{-r^2/2\beta^2}\,\mathrm{d}r, \tag{14}$$

being an $n \neq 1$ extension of equation (4).

[1] A number of typographical errors may be noted in section 6.6 of [5]. Near the middle of pp 219 and 220, $(n+m)!$ should be replaced by $(n+m-1)!$. Near the middle of p 220, $L_m^{(n-1)}$ should replace $L_\nu^{(n-1)}$ and $L_m^{(n-1)}(t)$ should replace $L_m^{(n-1)}(t^2)$ near the bottom in the second expression for $T^*(\tau, m)$. On this same page, [11] should replace [12].

4. New infinite sums and finite convolution series of Laguerre polynomials

The associated Laguerre polynomials form a positive-definite orthogonal polynomial sequence. The Sheffer-type exponential generating function of these polynomials [12] is a key feature of part of the theory of bosonic normal ordering and the quantum field theory of partitions [10].

A finite convolution sum such as [17, 28] $L_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} L_{n-k}^{\alpha}(x) L_k^{\beta}(y)$ is well known and is relevant to the corresponding combinatorial field theory. In this section, we develop new infinite sums and finite convolution series of the associated Laguerre polynomials. The main result is the following

Proposition 3. *For $\alpha > -1$ and k a positive integer, we have*

$$\sum_{j=1}^{n-1} \frac{L_{j-1}^{\alpha}(x) L_{n-j-1}^{k+1}(y)}{n-j} = \frac{k!}{y^{k+1}} \left[\sum_{\ell_1=0}^{k} \frac{y^{\ell_1}}{\ell_1!} L_{n-1}^{\alpha}(x) - \sum_{\ell_2=0}^{k} \frac{y_2^{\ell}}{\ell_2!} L_{n-1}^{\alpha+\ell_2}(x+y) \right]. \tag{15}$$

The proof of proposition 3 relies on two lemmas.

Lemma 1. *For $\alpha > -1$ and $|w| < 1$ we have*

$$\sum_{n=1}^{\infty} \frac{L_{n-1}^{\alpha}(x)}{n} w^n = \frac{e^x}{x^{\alpha}} \left[\Gamma(\alpha, x) - \Gamma\left(\alpha, \frac{x}{1-w}\right) \right]. \tag{16}$$

Lemma 2. *For j a nonnegative integer and $|w| < 1$ we have*

$$\sum_{n=1}^{\infty} \frac{L_{n-1}^{j+1}(x)}{n} w^n = \frac{j!}{x^{j+1}} \left[\sum_{m=0}^{j} \frac{x^m}{m!} - e^{xw/(w-1)} \sum_{m=0}^{j} \frac{1}{m!} \frac{x^m}{(1-w)^m} \right]. \tag{17}$$

Lemma 2 follows from lemma 1 due to the property [16] $\Gamma(n+1, x) = n!\, e^{-x} \sum_{m=0}^{n} x^m/m!$ when n is a nonnegative integer.

Proof of lemma 1. We integrate a standard generating function for the associated Laguerre polynomials [3, 16], $\sum_{n=1}^{\infty} L_n^{\alpha}(x) z^n = (1-z)^{-\alpha-1} \exp[xz/(z-1)]$ for $|z| < 1$. With multiple changes of variable we have [16]

$$\sum_{n=1}^{\infty} \frac{L_{n-1}^{\alpha}(x)}{n} w^n = -\int_0^{w/(w-1)} (1-v)^{\alpha-1} e^{xv}\, dv = e^x \int_1^{1/(1-w)} u^{\alpha-1} e^{-xu}\, du \tag{18a}$$

$$= \frac{e^x}{x^{\alpha}} \int_x^{x/(1-w)} \zeta^{\alpha-1} e^{-\zeta}\, d\zeta = \frac{e^x}{x^{\alpha}} \left[\Gamma(\alpha, x) - \Gamma\left(\alpha, \frac{x}{1-w}\right) \right]. \tag{18b}$$

Alternatively we may apply an integral representation for L_n^{α} in terms of the Bessel function J_{α} given as equation (6.2.15) on p 286 of [3]. Interchanging summation and integration then gives [16]

$$\sum_{n=1}^{\infty} \frac{L_{n-1}^{\alpha}(x)}{n} w^n = e^x x^{-\alpha/2} \int_0^{\infty} t^{\alpha/2-1} [e^{-(1-w)t} - e^{-t}] J_{\alpha}[2(tx)^{1/2}]\, dt$$

$$= e^x x^{-\alpha} \left[\gamma\left(\alpha, \frac{x}{1-w}\right) - \gamma(\alpha, x) \right], \tag{18c}$$

where $\gamma(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x)$. □

(Proof of proposition 3). We consider the product of series

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n \sum_{m=1}^{\infty} \frac{L_{m-1}^{\beta}(y)}{m} t^{m-1} = \frac{e^{xt/(t-1)}}{(1-t)^{\alpha+1}} \frac{e^y}{t y^{\beta}} \left[\Gamma(\beta, y) - \Gamma\left(\beta, \frac{y}{1-t}\right)\right], \tag{19}$$

where lemma 1 has been used. The sum on the left side may be transformed to

$$\sum_{n=0}^{\infty}\sum_{j=n}^{\infty} \frac{L_n^{\alpha}(x)}{j-n+1} L_{j-n}^{\beta}(y)t^j = \sum_{j=0}^{\infty}\sum_{n=1}^{j+1} \frac{L_{n-1}^{\alpha}(x)}{j-n+2} L_{j-n+1}^{\beta}(y)t^j. \tag{20}$$

By making use of the generating function of the associated Laguerre polynomials and applying lemma 2 to the right side of equation (19) we find

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n \sum_{m=1}^{\infty} \frac{L_{m-1}^{\beta}(y)}{m} t^{m-1} = \frac{k!}{y^{k+1}} \left[\sum_{n=0}^{\infty} L_{n+1}^{\alpha}(x)t^n \sum_{\ell_1=0}^{k} \frac{y^{\ell_1}}{\ell_1!} - \sum_{\ell_2=0}^{k} \frac{y^{\ell_2}}{\ell_2!} \sum_{n=0}^{\infty} L_{n+1}^{\alpha+\ell_2}(x+y)t^n\right]. \tag{21}$$

We then compare the coefficients of t^j on both sides of this equation. Putting $j \to j-2$ and reversing the roles of the indices j and n gives the statement of equation (15). □

Lemmas 1 and 2 and proposition 3 have very many special cases, and we mention some of these for illustration. Obviously from equation (18*a*), $\alpha = 1$ is a very special case, $\Gamma(1, x) = e^{-x}$, and we have

$$\sum_{n=1}^{\infty} \frac{L_{n-1}^{1}(x)}{n} w^n = \frac{1}{x}[1 - e^{xw/(w-1)}], \qquad |w| < 1, \tag{22}$$

and

$$\sum_{j=1}^{n-1} \frac{L_{j-1}^{1}(x)L_{n-j-1}^{1}(y)}{n-j} = \frac{1}{y}\left[L_{n-1}^{1}(x) - L_{n-1}^{1}(x+y)\right]. \tag{23}$$

When $\alpha = 1/2$ and $\alpha = 0$ the result of lemma 1 may be written in terms of the probability integral Φ and the exponential integral Ei [16], respectively:

$$\sum_{n=1}^{\infty} \frac{L_{n-1}^{1/2}(x)}{n} w^n = \sqrt{\frac{\pi}{x}} \left[\Phi\left(\sqrt{\frac{x}{1-w}}\right) - \Phi(\sqrt{x})\right], \tag{24a}$$

$$\sum_{n=1}^{\infty} \frac{L_{n-1}(x)}{n} w^n = e^x \left[\mathrm{Ei}\left(\frac{x}{w-1}\right) - \mathrm{Ei}(-x)\right]. \tag{24b}$$

Further taking the limit $w \to 1$ in lemma 1 or equation (24) yields very special cases often listed in the standard tables of Hansen, chapter 48 [17]. For instance, we have $\sum_{n=1}^{\infty} L_{n-1}(x)/n = -e^x \mathrm{Ei}(-x)$. Therefore, our results subsume many of the known Laguerre expansions in the literature.

Similar to how we obtained lemma 1, we may continue to integrate that relation. Omitting the details, we have found

Lemma 3. *For $\alpha > -1$ and $|z| < 1$ we have*

$$\sum_{n=2}^{\infty} \frac{L_{n-2}^{\alpha}(x)}{n(n-1)} z^n = \frac{e^x}{x^{\alpha}} \left\{(z-1)\Gamma(\alpha, x) + x\left[\Gamma(\alpha-1, x) - \Gamma\left(\alpha-1, \frac{x}{1-z}\right)\right]\right.$$
$$\left. + (1-z)\Gamma\left(\alpha, \frac{x}{1-z}\right)\right\}, \tag{25}$$

and

Lemma 4. *For $\alpha > -1$ and $|w| < 1$ we have*

$$\sum_{n=3}^{\infty} \frac{L_{n-3}^{\alpha}(x)}{n(n-1)(n-2)} w^n = \frac{e^x}{x^\alpha} \left\{ \left(\frac{1}{2} w - 1 \right) w \Gamma(\alpha, x) + x(w-1)\Gamma(\alpha - 1, x) \right.$$
$$+ (w-1)\Gamma\left(\alpha - 1, \frac{x}{1-w}\right) + \frac{x^2}{2}\left[\Gamma(\alpha - 2, x) - \Gamma\left(\alpha - 2, \frac{x}{1-w}\right)\right]$$
$$\left. + \frac{1}{2}\Gamma(\alpha, x) - \frac{1}{2}(w-1)^2\Gamma\left(\alpha, \frac{x}{1-w}\right)\right\}. \tag{26}$$

These and related expressions have potential applications in summing perturbation series in quantum mechanics [27]. Again lemmas 3 and 4 have a great many special cases. As an illustration of special cases of lemma 3 we have

$$\sum_{n=2}^{\infty} \frac{L_{n-2}^{\alpha}(x)}{n(n-1)} = e^x x^{1-\alpha} \Gamma(\alpha - 1, x), \tag{27a}$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{L_{n-2}^{\alpha}(x)}{n(n-1)} = \frac{e^x}{x^\alpha} \{-2\Gamma(\alpha, x) + x[\Gamma(\alpha - 1, x) - \Gamma(\alpha - 1, x/2)] - 2\Gamma(\alpha, x/2)\}, \tag{27b}$$

and

$$\sum_{n=2}^{\infty} (-1)^n \frac{L_{n-2}^{2}(x)}{n(n-1)} = \frac{1}{x^2}[-2 - x(1 + e^{x/2}) - 2\, e^{x/2}(1 + x/2)]. \tag{27c}$$

5. Summary and brief discussion

We have described the scale-space and quantum mechanical origin of shapelets. These complete and orthonormal basis functions are well suited to image processing tasks. Their abundant analytic properties permit compact expressions for convolution, other integrations, and linear transformations to be developed. Shapelets and related functions are very useful for the analytic formulation of multicentre integrals in variational calculations of molecular electronic wavefunctions [24]. Further research for using shapelets in a quantum computing environment is required. We have been able to develop new infinite sums and finite convolution series of associated Laguerre polynomials that generalize several known results.

Much of the theory we developed with generating functions for polar shapelets can be carried over directly for Cartesian shapelets. In addition, each set of Hermite and associated Laguerre polynomials has other existing generating functions. This permits generalization of many of our results and will be described elsewhere. The generating function phenomenology extends to applications in several areas of mathematical physics and image processing, including the combinatorics of bosonic quantum field theory in zero dimension.

The functions [19]

$$\Lambda_{n,\mu}^{\alpha}(x, t) \equiv \frac{L_n^{\alpha}(|\mu| x)}{L_n^{\alpha}(0)} e^{-|\mu| x/2 + i\mu t}, \qquad x \geqslant 0, \tag{28}$$

where μ and t are real, have yet more properties than the associated Laguerre polynomials themselves. The functions Λ have a positive convolution structure and a group theoretic interpretation as spherical functions [19]. On a certain space R^{2n+1} the convolution is commutative. Furthermore, there is the linearization formula

$$\Lambda^{\alpha}_{m,\mu}(x,0)\Lambda^{\alpha}_{n,\nu}(x,0) = \sum_{i=0}^{m+n} c_i \Lambda^{\alpha}_{m+n-i,\mu+\nu}(x,0), \tag{29}$$

where $c_i \geqslant 0$ if μ, $\nu \geqslant 0$ [4, 19]. Therefore, the functions $\Lambda^{\alpha}_{n,\mu}$ may be quite suitable for image processing applications.

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Appendix. A triple product integral for Cartesian shapelet convolution

Motivated by applications of convolution, deconvolution and image deprojection with Cartesian shapelets, we develop here a closed form for an infinite integral for a triple product of Hermite polynomials H_m [26]. We let

$$L_{\ell mn}(a,b,c) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_\ell(ax) H_m(bx) H_n(cx)\, \mathrm{e}^{-x^2}\, \mathrm{d}x. \tag{A.1}$$

Based upon a consideration of parity, this integral vanishes unless $\ell + m + n$ is an even integer. We may note that the case $a = b = c = 1$ is known in closed form [3, 16], as well as effectively the case $a = b = c = 1/\sqrt{2}$.

We have

Proposition A. *For ℓ, m and n nonnegative even integers,*

$$\begin{aligned} L_{\ell mn}(a,b,c) = \ell! m! n! &\sum_{\substack{q_1=0\\ \text{even}}}^{\min(\ell,m)} \sum_{\substack{q_2=0\\ \text{even}}}^{\min(\ell,n)} \sum_{\substack{q_3=0\\ \text{even}}}^{\min(m,n)} \\ &\times \frac{(a^2-1)^{(\ell-q_1-q_2)/2}(b^2-1)^{(m-q_1-q_3)/2}(c^2-1)^{(n-q_2-q_3)/2}}{[(m-q_1-q_3)/2]![(n-q_2-q_3)/2]![(n-q_2-q_3)/2]!} \frac{2^{q_1+q_2+q_3}}{q_1!q_2!q_3!} \\ &\times a^{q_1+q_2} b^{q_1+q_3} c^{q_2+q_3}. \end{aligned} \tag{A.2}$$

Proof of equation (A.2). We proceed as in [3] (p 318) for the case of $a = b = c = 1$, using the exponential generating function of the Hermite polynomials. We have

$$\begin{aligned} &\int_{-\infty}^{\infty} \sum_{\ell,m,n=0}^{\infty} \frac{1}{\ell!}\frac{1}{m!}\frac{1}{n!} H_\ell(ax)H_m(bx)H_n(cx)\,\mathrm{e}^{-x^2} r^\ell s^m t^n\,\mathrm{d}x \\ &\quad = \int_{-\infty}^{\infty} \exp(2axr - r^2)\exp(2bxs - s^2)\exp(2cxt - t^2)\,\mathrm{e}^{-x^2}\,\mathrm{d}x \\ &\quad = \exp(2(rs+rt+st)) \int_{-\infty}^{\infty} \exp(-(x-r-s-t)^2) \\ &\qquad \times \exp(2x[(a-1)r + (b-1)s + (c-1)t])\,\mathrm{d}x \\ &\quad = \sqrt{\pi}\exp((a^2-1)r^2 + (b^2-1)s^2 + (c^2-1)t^2 + 2ar(bs+ct) + 2bcst). \end{aligned} \tag{A.3}$$

We now expand the right side in powers of r, s and t,

$$\int_{-\infty}^{\infty} \sum_{\ell,m,n=0}^{\infty} \frac{1}{\ell!}\frac{1}{m!}\frac{1}{n!} H_\ell(ax) H_m(bx) H_n(cx)\, \mathrm{e}^{-x^2} r^\ell s^m t^n \,\mathrm{d}x$$

$$= \sqrt{\pi} \sum_{k_1,k_2,k_3=0}^{\infty} \sum_{q_1,q_2,q_3=0}^{\infty} \frac{(a^2-1)^{k_1}(b^2-1)^{k_2}(c^2-1)^{k_3}}{k_1!k_2!k_3!} \frac{2^{q_1+q_2+q_3}}{q_1!q_2!q_3!}$$
$$\times\, a^{q_1+q_2} b^{q_1+q_3} c^{q_2+q_3} r^{2k_1+q_1+q_2} s^{2k_2+q_1+q_3} t^{2k_3+q_2+q_3}$$

$$= \sqrt{\pi} \sum_{q_1,q_2,q_3=0}^{\infty} \sum_{\ell=q_1+q_2}^{\infty} \sum_{m=q_1+q_3}^{\infty} \sum_{n=q_2+q_3}^{\infty}$$
$$\times \frac{(a^2-1)^{(\ell-q_1-q_2)/2}(b^2-1)^{(m-q_1-q_3)/2}(c^2-1)^{(n-q_2-q_3)/2}}{[(m-q_1-q_3)/2]![(n-q_2-q_3)/2]![(n-q_2-q_3)/2]!} \frac{2^{q_1+q_2+q_3}}{q_1!q_2!q_3!}$$
$$\times\, a^{q_1+q_2} b^{q_1+q_3} c^{q_2+q_3} r^\ell s^m t^n, \tag{A.4}$$

where we changed the variables of summation. Upon reordering the sums on the right side of this equation and equating the coefficients of like powers of r, s and t there with those on the left side, we obtain the proposition.

Remarks. (i) The proof when one of the indices ℓ, m, or n is even and the other two are odd is very similar and is omitted. (ii) Presumably proposition A may also be obtained by the use of linearization formulae for the product of two Hermite polynomials. For the case of $L_{\ell,m,n}(a, b, b)$ this is obvious [3] (p 318). (iii) In the article 'Shapelets-II' of [26], recurrence relations were given for $L_{\ell,m,n}(a, b, c)$, based upon integration by parts. In the short table 1 given there, in fact only one entry is genuinely a triple product integral, since all others contain at least one index ℓ, m, or n that is zero. (iv) Wavelets ψ satisfy the admissibility condition $\int_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 0$ and it was very recently asked to find such even functions with weight factor $\exp(-x^2/2)$ generalizing the function $\psi_M(x) = \pi^{-1/4}\exp(-x^2/2)(1-x^2)$ [14]. Based upon the orthogonality properties of products of Hermite polynomials, our answer is immediate. We may take ψ, a 'mother wavelet', as any of the functions, omitting a constant multiplier, $H_n(x/\sqrt{2})\exp(-x^2/2)$ or $H_n(x/\sqrt{2})H_m(x/\sqrt{2})\exp(-x^2/2)$, where the subscripts are even integers, and $n > 0$, $n \neq m$, respectively. More generally, we may take any linear combination of these functions as an admissible function. We recognize the Mexican hat wavelet as $\psi_M(x) \propto \exp(-x^2/2)H_2(x/\sqrt{2})$. The elaborate use of Dirac formalism [14] has not been required to generalize it. In essence, we have recovered in very short order all of the observations of example 1 of [30].

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